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Abstract

Variable annuities, as a class of retirement income products, allow equity market exposure for a policyholder's retirement fund with electable additional guarantees to limit the downside risk of the market. Management fees and guarantee insurance fees are charged respectively for the market exposure and for the protection from the downside risk. We investigate the pricing of variable annuity guarantees under optimal withdrawal strategies when management fees are present. We consider from both policyholder's and insurer's perspectives optimal withdrawal strategies and calculate the respective fair insurance fees. We reveal a discrepancy where the fees from the insurer's perspective can be significantly higher due to the management fees serving as a form of market friction. Our results provide a possible explanation of lower guarantee insurance fees observed in the market than those predicted from the insurer's perspective. Numerical experiments are conducted to illustrate the results.

JEL classification: C61, G22

Keywords: pricing; variable annuity guarantees; management fees; dynamic programming

1. Introduction

Variable annuities (VA) with guarantees of living and death benefits are offered by wealth management and insurance companies worldwide to assist individuals in managing their pre-retirement and post-retirement financial plans. These products take advantages of market growth while provide a protection of the savings against market downturns. Similar guarantees are also available for life insurance policies (Bacinello and Ortu [2]). The VA contract cash flows received by the policyholder are linked to the investment portfolio choice and performance (e.g. the choice of mutual fund and its strategy) while traditional annuities provide

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a pre-defined income stream in exchange for a lump sum payment. Holders of VA policies are required to pay management fees regularly during the term of the contract for the wealth management services.

A variety of VA guarantees, also known as VA riders, can be elected by policyholders at the cost of additional insurance fees. Common examples of VA guarantees include guaranteed minimum accumulation benefit (GMAB), guaranteed minimum withdrawal benefit (GMWB), guaranteed minimum income benefit (GMIB) and guaranteed minimum death benefit (GMDB), as well as a combination of them, e.g., guaranteed minimum withdrawal and death benefit (GMWDB), among others. These guarantees, generically denoted as GMxB, provide different types of protection against market downturns, shortfall of savings due to longevity risk or assurance of stability of income streams. Precise specifications of these products can vary across categories and issuers. See Bauer et al. [3], Ledlie et al. [20], Kalberer and Ravindran [17] for an overview of these products.

The Global Financial Crisis during 2007-08 led to lasting adverse market conditions such as low interest rates and asset returns as well as high volatilities for VA providers. Under these conditions, the VA guarantees become more valuable, and the fulfillment of the corresponding required liabilities become more demanding. The post-crisis market conditions have called for effective hedging of risks associated with the VA guarantees (Sun et al. [29]). As a consequence, the need for accurate estimation of hedging costs of VA guarantees has become increasingly important. Such estimations consist of risk-neutral pricing of future cash flows that must be paid by the insurer to the policyholder in order to fulfill the liabilities of the VA guarantees.

There have been a number of contributions in the academic literature considering the pricing of VA guarantees. A range of numerical methods are considered, including standard and regression-based Monte Carlo (Huang and Kwok [15]), partial differential equation (PDE) and direct integration methods (Milevsky and Salisbury [23], Dai et al. [9], Chen and Forsyth [5], Bauer et al. [3], Luo and Shevchenko [21, 22], Forsyth and Vetzal [11], Shevchenko and Luo [28]). A comprehensive overview of numerical methods for the pricing of VA guarantees is provided in Shevchenko and Luo [27].

In this article we focus on GMWDB which provides a guaranteed withdrawal amount per year until the maturity of the contract regardless of the investment performance, as well as a lump-sum of death benefit in case the policyholder dies over the contract period. The guaranteed withdrawal amount is determined such that the initial investment is returned over the life of the contract. The death benefit may assume different forms depending on the details of the contract. When pricing GMWDB, one typically assumes either a pre-determined (static) policyholder behavior in withdrawal and surrender, or an active (dynamic) strategy where the policyholder "optimally" decides the amount of withdrawal at each withdrawal date depending on the information available at that date.

One of the most debated aspects in the pricing of GMWDB with active withdrawal strategies is the policyholders' withdrawal behaviors (Cramer et al. [7], Chen and Forsyth [5], Moenig and Bauer [24], Forsyth and Vetzal [11]). It is often customary to refer to the withdrawal strategy that maximizes the expected liability, or the hedging cost, of the VA guarantee as the "optimal" strategy. Even though such a strategy underlies the worst case scenario for the VA provider with the highest hedging cost, it may not coincide with the real-world behavior of the policyholder. Nevertheless, the price of the guarantee under this strategy provides an upper bound of hedging cost from the insurer's perspective, which is often referred to as the "value" of the guarantee. The real-world behaviors of policyholders often deviate from this "optimal" strategy, as is noted in Moenig and Bauer [24]. Different models have been proposed to account for the real-world behaviors of policyholders, including the reduced-form exercise rules of Ho et al. [14], and the subjective risk neutral valuation approach taken by Moenig and Bauer [24]. In particular, it is concluded by Moenig and Bauer [24] that a subjective risk-neutral valuation methodology that takes different tax structures into consideration is in line with the corresponding findings from empirical observations.

Similar to the tax consideration in Moenig and Bauer [24], the management fee is a form of market friction that would affect policyholders' rational behaviors. However, such effects of management fees have not been considered in the VA pricing literature. When the management fee is zero and deterministic withdrawal behavior is assumed, Hyndman and Wenger [16] and Fung et al. [12] show that risk-neutral pricing of guaranteed withdrawal benefits in both a policyholder's and an insurer's perspectives will result in the same fair insurance fee. Few studies that take management fees into account in the pricing of VA guarantees include Bélanger et al. [4], Chen et al. [6] and Kling et al. [18]. In these studies, fair insurance fees are considered from the insurer's perspective with the management fees as given. The important question of how the management fees as a form of market friction will impact withdrawal behaviors of the policyholder, and hence the hedging cost for the insurer, is yet to be examined in a dynamic withdrawal setting. The main goal of the paper is to address this question.

The paper contributes to the literature in three aspects. First, we consider two pricing approaches based on the policyholder's and the insurer's perspective. In the literature it is most often the case that only an insurer's perspective is considered, which might result in mis-characterisation of the policyholder's withdrawal strategies. Second, we characterize the impact of management fees on the pricing of GMWDB, and demonstrate that the two afore-mentioned pricing perspectives lead to different fair insurance fees due to the presence of management fees. In particular, the fair insurance fees from the policyholder's perspective is lower than those from the insurer's perspective. This provides a possible justification of lower insurance fees observed in the market. Third, the sensitivity of the fair insurance fees to management fees under different market conditions and contract parameters are investigated and quantified through numerical examples.

The paper is organized as follows. In Section 2 we present the contract details of the GMWDB guarantee together with its pricing formulation under a stochastic optimal control framework. Section 3 derives the policyholder's value function under the risk-neutral pricing approach, followed by the insurer's net liability function in Section 4. In Section 5 we compare the two withdrawal strategies that maximize the policyholder's value and the insurer's liability, respectively, and discuss the role of the management fees in their relations. Section 6 demonstrates our approaches via numerical examples. Section 7 concludes with remarks and discussion.

2. Formulation of the GMWDB pricing problem

We begin with the setup of the framework for the pricing of GMWDB and describe the features of this type of guarantees. The problem is formulated under a general setting so that the resulting pricing formulation can be applied to different GMWDB contract specifications. Besides the general setting, we also consider a very specific simple GMWDB contract, which will be subsequently used for illustration purposes in numerical experiments presented in Section 6.

The VA policyholder's retirement fund is usually invested in a managed wealth account that is exposed to financial market risks. A management fee is usually charged for this investment service. In addition, if GMWDB is elected, extra insurance fees will be charged for the protection offered by the guarantee provider (insurer). We assume the wealth account guaranteed by the GMWDB is subject to continuously charged proportional management fees independent of any fees charged for the guarantee insurance. The purpose of these management fees is to compensate for the wealth management services provided, or perhapse merely for the access to the guarantee insurance on the investment. This fee should not be confused with other forms of market frictions, e.g., trasaction costs, if any, that must incur when tracking a given equity index. Given the proliferation of index-tracking exchange-traded funds in recent years, with much desired liquidity at a fraction of the costs of the conventional index mutual funds, see, e.g., [1, 19, 26], regarding these management fees as additional costs to policyholder beyond the normal market frictions seems to be a reasonable assumption.

The hedging cost of the guarantee, on the other hand, is paid by proportional insurance fees continuously charged to the wealth account. The fair insurance fee rate, or the fair fee in short, refers to the minimal insurance fee rate required to fund the hedging portfolio, so that the guarantee provider can eliminate the market risk associated with the selling of the guarantees.

We consider the situation where a policyholder purchases the GMWDB rider in order to protect his wealth account that tracks an equity index $S(t)$ at time $t \in [0, T]$, where 0 and T correspond to the inception and expiry dates. The equity index account is modelled under the risk-neutral probability measure $\mathbb{Q}$ following the stochastic differential equation (SDE)

$$dS(t) = S(t)\left(r(t)dt + \sigma(t)dB(t)\right), \quad t \in [0, T], \tag{1}$$

where $r(t)$ is the risk-free short interest rate, $\sigma(t)$ is the volatility of the index, which are time-dependent and can be stochastic, and $B(t)$ is a standard $\mathbb{Q}$-Brownian motion modelling the uncertainty of the index. Here, we follow standard practices in the literature of VA guarantee pricing by modelling under the risk-neutral probability measure $\mathbb{Q}$, which allows the pricing of stochastic cash flows to be given as the risk-neutral expectation of the discounted cash flows. The risk-neutral probability measure $\mathbb{Q}$ exists if the underlying financial market satisfies certain "no-arbitrage" conditions. Adopting risk-neutral pricing here assumes that stochastic cash flows in the future can be replicated by dynamic hedging without transaction fees. For details on risk-neutral pricing and the underlying assumptions, see, e.g., Delbaen and Schachermayer [10] for an account under very general settings.

The wealth account $W(t), t \in [0, T]$ over the lifetime of the GMWDB contract is invested into the index S, subject to management fees charged by a wealth manager at the rate $\alpha_{\mathrm{m}}(t)$. An additional charge of insurance fees at rate $\alpha_{\mathrm{ins}}(t)$ for the GMWDB rider is collected by the insurer to pay for the hedging cost of the guarantee. Both fees are deterministic, time-dependent and continuously charged. Discrete fees may be modelled similarly without any difficulty. The wealth account in turn evolves as

$$dW(t) = W(t)\left((r(t) - \alpha_{\mathrm{tot}}(t))dt + \sigma(t)dB(t)\right), \tag{2}$$

for any $t \in [0, T]$ at which no withdrawal of wealth is made. Here, $\alpha_{\mathrm{tot}}(t) = \alpha_{\mathrm{ins}}(t) + \alpha_{\mathrm{m}}(t)$ is the total fee rate. The GMWDB contract allows the policyholder to withdraw from a guarantee account $A(t), t \in [0, T]$ on a sequence of pre-determined contract event dates, $0 = t_0 < t_1 < \cdots < t_N = T$. The initial guarantee $A(0)$ usually matches the initial wealth $W(0)$. The guarantee account stays constant unless a withdrawal is made on one of the event dates, which changes the guarantee account balance. If the policyholder dies on or before the maturity T, the death benefit will be paid at the next event date immediately following the death of the policyholder. Additional features such as early surrender can be included straightforwardly but will not be considered in this article to avoid unnecessary complexities.

To simplify notations, we denote by $\mathbf{Y}(t)$ the vector of state variables at t, given by

$$\mathbf{Y}(t) = (r(t), \sigma(t), S(t), W(t), A(t)), \quad t \in [0, T]. \tag{3}$$

Here, we assume that all state variables follow Markov processes under the risk-neutral probability measure $\mathbb{Q}$, so that $\mathbf{Y}(t)$ contains all the market and account balances information available at t. For simplicity, we assume the state variables $r(t)$, $\sigma(t)$ and $S(t)$ are continuous, and $W(t)$ and $A(t)$ are left continuous with right limit (LCRL). We include the index value $S(t)$ in $\mathbf{Y}(t)$, which under the current model may seem redundant, due to the scale-invariance of the geometric Brownian motion type model (1). In general, however, $S(t)$ may determine the future dynamics of S in a nonlinear fashion, as is the case under, e.g., the minimal market model described in Platen and Heath [25].

We define $I(t), t \in [0,T]$ as the life indicator function of an individual policyholder as the following: $I(t) = 1$ if the policyholder was alive on the last event date on or before t; $I(t) = 0$ if the policyholder was alive on the second-to-the-last event date prior to t but died on or before the last event date; $I(t) = -1$ if the policyholder died on or before the second-to-the-last event date prior to t. We assume the policyholder is alive at t_0. The life indicator function $I(t)$ therefore starts at $I(t_0) = 1$, is right continuous with left limit (RCLL), and remains constant between two consecutive event dates. Note that mortality information contained in $I(t)$ (RCLL) comes before any jumps of the LCRL account balances $W(t)$ and $A(t)$ on the event dates, reflecting the situation that any jumps in these account balances may depend on the mortality information. We denote the vector of state variables including $I(t)$ as $\mathbf{X}(t) = (\mathbf{Y}(t)^\top, I(t))^\top$, and we denote by $\mathrm{E}_t^{\mathbb{Q}}[\cdot]$ the risk-neutral expectation conditional on the state variables $\mathbf{X}(t)$ at t, i.e., $\mathrm{E}_t^{\mathbb{Q}}[\cdot] := \mathrm{E}^{\mathbb{Q}}[\cdot|\mathbf{X}(t)]$. Note that the risk-neutral measure $\mathbb{Q}$ is assumed to extend to the mortality risk represented by the life indicator $I(t)$.

On event dates $t_n, n = 1, \ldots, N$, a nomimal withdrawal γ_n from the guarantee account is made. The policyholder, if alive, may choose γ_n on $t_n < T$. Otherwise a liquidation withdrawal of $\max(W(t_n), A(t_n))$ is made. That is,

$$\gamma_n = \Gamma(t_n, \mathbf{Y}(t_n)) \mathbb{1}_{\{I(t_n)=1, n<N\}} + \max\left(W(t_n), A(t_n)\right) \mathbb{1}_{\{I(t_n)=0 \,\mathrm{or}\, n=N\}}, \tag{4}$$

where $\mathbb{1}_{\{\}}$ denotes the indicator function of an event, and $\Gamma(\cdot,\cdot)$ is referred to as the *withdrawal strategy* of the policyholder. The real cash flow received by the policyholder, which may differ from the nominal amount, is denoted by $C_n(\gamma_n, \mathbf{X}(t_n))$. This is given by

$$C_n(\gamma_n, \mathbf{X}(t_n)) = P_n(\gamma_n, \mathbf{Y}(t_n)) \mathbb{1}_{\{I(t_n)=1\}} + D_n(\mathbf{Y}(t_n)) \mathbb{1}_{\{I(t_n)=0\}}, \tag{5}$$

where $P_n(\gamma_n, \mathbf{Y}(t_n))$ is the payment received if the policyholder is alive, and $D_n(\mathbf{Y}(t_n))$ denotes the death benefit if the policyholder died during the last period. As a specific example, $P_n(\gamma_n, \mathbf{Y}(t_n))$ may be given by

$$P_n(\gamma_n, \mathbf{Y}(t_n)) = \gamma_n - \beta \max(\min(\gamma_n, A(t_n)) - G_n, 0). \tag{6}$$

Here the contractual withdrawal G_n is a pre-determined withdrawal amount specified in the GMWDB contract, and β is the penalty rate applied to the part of the withdrawal from the guarantee account exceeding the contractual withdrawal G_n. The $\min(\gamma_n, A(t_n))$ term in (6) accommodates the situation that at expiration of the contract, both accounts are liquidated, but only the guarantee account withdrawals exceeding the contractual rate G_n will be penalized. Excess balance on the wealth account after the guaranteed withdrawal is not subject to this penalty. An example of the death benefit may simply be taken as the total withdrawal without penalty, i.e.,

$$D_n(\mathbf{Y}(t_n)) = \max\left(W(t_n), A(t_n)\right). \tag{7}$$

Upon withdrawal by the policyholder, the guarantee account is changed by the amount $J_n(\gamma_n, \mathbf{Y}(t_n))$, that is,

$$A(t_n^+) = A(t_n) - J_n(\gamma_n, \mathbf{Y}(t_n)), \tag{8}$$

where $A(t_n^+)$ denotes the guarantee account balance "immediately after" the withdrawal. For example, $J_n(\gamma_n, \mathbf{Y}(t_n))$ may be given by

$$J_n(\gamma_n, \mathbf{Y}(t_n)) = \gamma_n \mathbb{1}_{\{I(t_n)=1\}} + A(t_n)\mathbb{1}_{\{I(t_n)\leq 0\}}, \tag{9}$$

i.e., the guarantee account balance is reduced by the withdrawal amount if the policyholder is alive and the policy has not expired. Otherwise the account is liquidated. The guarantee account stays nonnegative, that is, γ_n if chosen by the policyholder must be such that $J_n(\gamma_n, \mathbf{Y}(t_n)) \leq A(t_n)$. The wealth account is reduced by the amount γ_n upon withdrawal and remains nonnegative. That is,

$$W(t_n^+) = \max(W(t_n) - \gamma_n, 0), \tag{10}$$

where $W(t_n^+)$ is the wealth account balance immediately after the withdrawal. It is assumed that $\gamma_0 = 0$, i.e., no withdrawals at the start of the contract. Both the wealth and the guarantee account balance are 0 after contract expiration. That is

$$W(T^+) = A(T^+) = 0. \tag{11}$$

The policyholder's value function at time t is denoted by $V(t, \mathbf{X}(t)), t \in [0, T]$, which corresponds to the risk-neutral value of all cash flows to the policyholder on or after time t. The remaining policy value after the final cash flow is thus 0, i.e.,

$$V(T^+, \mathbf{X}(T^+)) = 0. \tag{12}$$

3. Calculating the Policyholder's Value Function

We now calculate the policyholder's value function $V(t, \mathbf{X}(t))$ as the risk-neutral expected value of policyholder's future cash flows at time $t \in [0, T]$. The risk-neutral valuation of the policyholder's future cash flows can be regarded as the value of the remaining term of the VA contract from the poliyholer's perspective. As mentioned in the beginning of Section 2, valuation under the risk-neutral pricing approach assumes that the cash flows may be replicated by hedging portfolios without market frictions. This may be carried out by a third-party independent agent, if not directly by the individual policyholder. In Section 5, we describe a scenario where a third-party agent with access to frictionless hedging strategies of the policyholder's cash flows may earn a profit from the policyholder's sub-optimal withdrawals.

Following Section 2, the policyholder's value function on an event date t_n can be written as

$$V(t_n, \mathbf{X}(t_n)) = C_n(\gamma_n, \mathbf{X}(t_n)) + V(t_n^+, \mathbf{X}(t_n^+)), \tag{13}$$

which by (5) can be further written as

$$V(t_n, \mathbf{X}(t_n)) = (P_n(\gamma_n, \mathbf{Y}(t_n)) + V_n(t_n^+, \mathbf{X}(t_n^+)))\mathbb{1}_{\{I(t_n)=1\}} + D_n(\mathbf{Y}(t_n))\mathbb{1}_{\{I(t_n)=0\}}, \tag{14}$$

since if the policyholder died during last period, the death benefit is the only cash flow to receive. Taking the risk-neutral expectation $\mathrm{E}^{\mathbb{Q}}_{t_n^-}[\cdot]$, we obtain the jump condition

$$V(t_n^-, \mathbf{X}(t_n^-)) = \Big((1-q_n)\Big(P_n(\gamma_n, \mathbf{Y}(t_n)) + V(t_n^+, \mathbf{X}(t_n^+))\Big) + q_n D_n(\mathbf{Y}(t_n))\Big)\mathbb{1}_{\{I(t_n^-)=1\}}, \tag{15}$$

where q_n is the risk-neutral probability that the policyholder died over (t_{n-1}, t_n), given that he is alive on the last withdrawal date t_{n-1}. That is,

$$q_n = \mathbb{Q}[I(t_n) = 0 | I(t_n^-) = 1]. \tag{16}$$

Here, we assume that the mortality risk is independent of the market risk under the risk-neutral probability measure. Under the assumption that the mortality risk is completely diversifiable, the risk-neutral mortality rate may be indentified with that under the real-world probability measure and inferred from a historical life table. Also note that the mortality information over $(t_{n-1}, t_n]$ is revealed at t_n, thus at t_n^- such information is not yet available. This assumption is not a model constraint since all decisions are made only on event dates.

The policy value at $t \in (t_{n-1}, t_n)$ is given by the expected discounted future policy value under the risk-neutral probability measure, given by

$$V(t, \mathbf{X}(t)) = \mathrm{E}^{\mathbb{Q}}_t\Big[e^{-\int_t^{t_n} r(s)ds} V(t_n^-, \mathbf{X}(t_n^-))\Big], \tag{17}$$

where $e^{-\int_t^{t_n} r(s)ds}$ is the discount factor. The initial policy value, given by $V(0, \mathbf{X}(0))$, can be calculated backward in time starting from the terminal condtion (12), using (15) and (17), as described in Algorithm 1.

As an illustrative example, we assume $r(t) \equiv r$, $\sigma(t) \equiv \sigma$ and $\alpha_{\text{tot}}(t) \equiv \alpha_{\text{tot}}$ as constants, and thus $V(t, \mathbf{X}(t)) = V(t, W(t))$ for $t \in (t_{n-1}, t_n)$. We now derive the partial differential equation (PDE) satisfied by the value function V through a hedging argument. We consider a delta hedging portfolio that, at time $t \in (t_{n-1}, t_n)$, takes a long position of the value function V and a short position of $\frac{W(t)\partial_W V(t,W(t))}{S(t)}$ shares of the index S. Here $\partial_W V(t, W(t)) \equiv \frac{\partial V(t,W)}{\partial W}|_{W=W(t)}$ is the partial derivative of $V(t, W)$ with respect to the second argument, evaluated at $W(t)$. Denoting the total value of this portfolio at $t \in (t_{n-1}, t_n)$ as $\Pi^V(t)$, the value of the delta hedging portfolio is given by

$$\Pi^V(t) = V(t, W(t)) - W(t)\partial_W V(t, W(t)). \tag{18}$$

By Ito's formula and (1), the SDE for Π^V can be obtained as

$$d\Pi^V(t) = \left(\partial_t V(t, W(t)) - \alpha_{\text{tot}} W(t)\partial_W V(t, W(t)) + \tfrac{1}{2}\sigma^2 W(t)^2 \partial_{WW} V(t, W(t))\right) dt, \tag{19}$$

for $t \in (t_{n-1}, t_n)$. Since the hedging portfolio Π^V is locally riskless, it must grow at the risk-free rate r, that is $d\Pi^V(t) = r\Pi^V(t)dt$. This along with (18) implies that the PDE satisfied by the value function $V(t, W)$ is given by

$$\partial_t V - rV + (r - \alpha_{\text{tot}})W\partial_W V + \frac{1}{2}\sigma^2 W^2 \partial_{WW} V = 0, \tag{20}$$

for $t \in (t_{n-1}, t_n)$ and $n = 1, \ldots, N$. The boundary conditions at t_n are specified by (12) and (15). The valuation formula (17) or the PDE (20) may be solved recursively by following Algorithm 1 to compute the initial policy value $V(0, \mathbf{Y}(0))$. It should be noted that (17) is general, and does not depend on the simplifying assumptions made in the PDE derivation.

Algorithm 1 Recursive computation of $V(0, \mathbf{X}(0))$

choose a withdrawal strategy Γ
initialize $V(T^+, \mathbf{X}(T^+)) = 0$
set $n = N$
while $n > 0$ **do**
  compute the withdrawal amount γ_n by by (4)
  compute $V(t_n^-, \mathbf{X}(t_n^-))$ by applying jump condition (15) with appropriate cash flows
  compute $V(t_{n-1}^+, \mathbf{X}(t_{n-1}^+))$ by solving (17) or (20) with terminal condition $V(t_n^-, \mathbf{X}(t_n^-))$
  $n = n - 1$
end while
return $V(0, \mathbf{X}(0)) = V(0^+, \mathbf{X}(0^+))$

4. Calculating the Insurer's Liability Function

The GMWDB contract may be considered from the insurer's perspective by examining the insurer's liabilities, given by the risk-neutral value of the cash flows that must be paid by the insurer in order to fulfill the GMWDB contract. We define the *net* liability function $L(t, \mathbf{X}(t)), t \in [0, T]$ as the time-t risk-neutral value of all future payments on or after t made to the policyholder by the insurer, less that of all insurance fee incomes over the same period.

The insurance fees, charged at the rate $\alpha_{\text{ins}}(t)$, $t \in [0, T]$, is called fair if the total fees exactly compensate for the insurer's total liability, such that the net liability is zero at time $t = 0$. That is,

$$L(0, \mathbf{X}(0)) = 0. \tag{21}$$

If $\alpha_{\text{ins}}(t) \equiv \alpha_{\text{ins}}$ is a constant, its value can be found by solving (21). The fair insurance fees represent the hedging cost for the insurer to deliver the GMWDB guarantee to the policyholder, which is often regarded as the value of the GMWDB rider, at least from the insurer's perspective. We emphasize here that this value may not be equal to the added value of the GMWDB rider to the policyholder's wealth account, as we will show in Section 5.

On an event date t_n, the actual cash flow received by the policyholder is given by (5). This cash flow is first paid out of the policyholder's real withdrawal from the wealth account, which is equal to $\min(W(t_n), \gamma_n)$, the smaller of the nominal withdrawal and the available wealth. If the wealth account has an insufficient balance, the rest must be paid by the insurer. If the real withdrawal exceeds the cash flow entitled to the policyholder, the insurer keeps the surplus. The payment made by the insurer at t_n is thus given by

$$c_n(\gamma_n, \mathbf{X}(t_n)) = C_n(\gamma_n, \mathbf{X}(t_n)) - \min(W(t_n), \gamma_n). \tag{22}$$

To compute $L(t, \mathbf{X}(t))$ for all $t \in [0, T]$ we first note that at maturity T, the terminal condition on L is given by

$$L(T^+, \mathbf{X}(T^+)) = 0, \tag{23}$$

i.e., no further liability or insurance fee income after maturity. Analogous to (13) and (15), the jump condition of L on event date t_n is given by

$$L(t_n, \mathbf{X}(t_n)) = c_n(\gamma_n, \mathbf{X}(t_n)) + L(t_n^+, \mathbf{X}(t_n^+)) \tag{24}$$

and

$$L(t_n^-, \mathbf{X}(t_n^-)) = \Big((1 - q_n)\Big(p_n(\gamma_n, \mathbf{Y}(t_n)) + L(t_n^+, \mathbf{Y}(t_n^+))\Big) + q_n d_n(\mathbf{Y}(t_n))\Big)\mathbb{1}_{\{I(t_n^-)=1\}}, \tag{25}$$

where $p_n(\gamma_n, \mathbf{Y}(t_n))$ and $d_n(\mathbf{Y}(t_n))$ refer to the insurance payments under $I(t_n) = 1$ and

$I(t_n) = 0$, respectively. See (22), (5) and (4).

At $t \in (t_{n-1}, t_n)$, the net liability function is given by the risk-neutral value of the remaining liabilities at t_n^- less any insurance fee incomes over the period (t, t_n), discounted at the risk-free rate. Specifically, we have

$$L(t, \mathbf{X}(t)) = \mathrm{E}_t^{\mathbb{Q}}\Big[e^{-\int_t^{t_n} r(s)ds} L(t_n^-, \mathbf{X}(t_n^-))\Big] - \mathrm{E}_t^{\mathbb{Q}}\Big[\int_t^{t_n} e^{-\int_t^s r(u)du} \alpha_{\mathrm{ins}}(s) W(s) ds\Big]. \tag{26}$$

Note that the net liability at t is reduced by expecting to receive insurance fees over time. Since this reduction decreases with time, the net liability *increases* with time over (t_{n-1}, t_n).

To give an example, we again assume constant $r(t) \equiv r$, $\sigma(t) \equiv \sigma$, $\alpha_{\mathrm{ins}}(t) \equiv \alpha_{\mathrm{ins}}$, $\alpha_{\mathrm{tot}}(t) \equiv \alpha_{\mathrm{tot}}$. Under these simplifying assumptions we have $L(t, \mathbf{X}(t)) = L(t, W(t))$ for $t \in (t_{n-1}, t_n)$. To derive the PDE satisfied by $L(t, W)$, consider a delta hedging portfolio that, at time $t \in (t_{n-1}, t_n)$, consists of a long position in the net liability function L and a short position of $\frac{W(t)\partial_W L(t,W(t))}{S(t)}$ shares of the index S. The value of the delta hedging portfolio, denoted as $\Pi^L(t)$, is given by

$$\Pi^L(t) = L(t, W(t)) - W(t)\partial_W L(t, W(t)). \tag{27}$$

By Ito's formula and (1), we obtain the SDE for Π^L as

$$d\Pi^L(t) = \big(\partial_t L(t, W(t)) - \alpha_{\mathrm{tot}} W(t)\partial_W L(t, W(t)) + \tfrac{1}{2}\sigma^2 W(t)^2 \partial_{WW} L(t, W(t))\big)\, dt, \tag{28}$$

where $t \in (t_{n-1}, t_n)$. Since the hedging portfolio Π^L is locally riskless and must grow at the risk-free rate r, as well as increase with the insurance fee income at rate $\alpha_{\mathrm{ins}} W(t)$ (see remarks after (26)), we must also have $d\Pi^L(t) = \big(r\Pi^L(t) + \alpha_{\mathrm{ins}} W(t)\big)\, dt$. This along with (27) implies that the PDE satisfied by the value function $L(t, W)$ is given by

$$\partial_t L - \alpha_{\mathrm{ins}} W - rL + (r - \alpha_{\mathrm{tot}}) W \partial_W L + \frac{1}{2}\sigma^2 W^2 \partial_{WW} L = 0, \tag{29}$$

for $t \in (t_{n-1}, t_n)$. The initial net liability can thus be computed by recursively solving (26) or (29) from terminal and jump conditions (23) and (25), as described in Algorithm 2.

5. The Wealth Manager's Value Function and Optimal Withdrawals

In the previous sections, the withdrawal strategy Γ has been assumed to be given. The withdrawal strategy serves as a control sequence affecting the policyholder's value function and the insurer's liability function. These withdrawals may thus be chosen to maximize either of these functions, leading to two distinct withdrawal strategies. In this section we formulate these two strategies and discuss their relations. In particular, we identify the wealth manager's value function that connects the two perspectives and the implications.

Algorithm 2 Recursive computation of $L(0, \mathbf{X}(0))$

choose a withdrawal strategy Γ
initialize $L(T^+, \mathbf{X}(T^+)) = 0$
set $n = N$
while $n > 0$ **do**
  compute the withdrawal amount γ_n by (4)
  compute $L(t_{n-1}^-, \mathbf{X}(t_{n-1}^-))$ by applying jump condition (25) with appropriate cash flows
  compute $L(t_{n-1}^+, \mathbf{X}(t_{n-1}^+))$ by solving (26) or (29) with terminal condition $L(t_n^-, \mathbf{X}(t_n^-))$
  $n = n - 1$
end while
return $L(0, \mathbf{X}(0)) = L(0^+, \mathbf{X}(0^+))$

5.1. The wealth manager's value function

We establish the relationship between the policy value V and the net liability L by defining the process

$$M(t, \mathbf{X}(t)) := L(t, \mathbf{X}(t)) + W(t) - V(t, \mathbf{X}(t)), \tag{30}$$

for $t \in [0, T]$. From (11) and (12) we obtain

$$M(T^+, \mathbf{X}(T^+)) = 0, \tag{31}$$

as the terminal condition for M. The jump condition for M can be obtained from (10), (13), (24) and (22) as

$$M(t_n, X(t_n)) = M(t_n^+, \mathbf{X}(t_n^+))), \tag{32}$$

and further more,

$$M(t_n^-, X(t_n^-)) = (1 - q_{n-1}) M(t_n^+, \mathbf{X}(t_n^+))) \mathbb{1}_{\{I(t_n^-)=1\}}. \tag{33}$$

From (17) and (26) we find the recursive relation for M as,

$$\begin{aligned} M(t, \mathbf{X}(t)) = \mathrm{E}_t^{\mathbb{Q}} & \Big[e^{-\int_t^{t_n} r(s)ds} M(t_n^-, \mathbf{X}(t_n^-)) \Big] \\ & + W(t) - \mathrm{E}_t^{\mathbb{Q}} \Big[e^{-\int_t^{t_n} r(s)ds} W(t_n) \Big] \\ & - \mathrm{E}_t^{\mathbb{Q}} \Big[\int_t^{t_n} e^{-\int_t^s r(u)du} \alpha_{\text{ins}}(s) W(s) ds \Big]. \end{aligned} \tag{34}$$

Note that the second and third lines in (34) can be identified with the time-t risk-neutral value of management fees over (t, t_n). To see this, we first note that the difference of the two terms in the second line is the time-t risk-neutral value of the total fees charged on the wealth account over (t, t_n), and the expectation in the third line is the time-t risk-neutral value of the insurance fees over the same period.

In lights of (31), (33) and (34), the process $M(t, \mathbf{X}(t)), t \in [0, T]$ defined by (30) is precisely the time-t risk-neutral value of future management fees, or *wealth manager's value function*. From (30), the policy value may be written as

$$V(t, \mathbf{X}(t)) = W(t) + L(t, \mathbf{X}(t)) - M(t, \mathbf{X}(t)), \tag{35}$$

i.e., the sum of the wealth and the value of the GMWDB rider, less the the wealth manager's value. At $t = 0$, this gives

$$V(0, \mathbf{X}(0)) = W(0) + L(0, \mathbf{X}(0)) - M(0, \mathbf{X}(0)). \tag{36}$$

Therefore, a withdrawal strategy that maximizes the insurer's total net liability $L(0, \mathbf{X}(0))$ in general is sub-optimal in maximizing the policyholder's total value $V(0, \mathbf{X}(0))$, since the wealth manager's total value $M(0, \mathbf{X}(0))$ depends on the withdrawals. The fair fee condition (21) becomes

$$V(0, \mathbf{X}(0)) = W(0) - M(0, \mathbf{X}(0)), \tag{37}$$

as in contrast to the $V(0) = W(0)$ condition often seen in the literature, when no management fees are charged.

5.2. Formulation of two optimization problems

We first formulate the policyholder's value maximization problem, i.e., maximizing the initial policy value $V(0, \mathbf{X}(0))$ by optimally choosing the sequence γ_n under $I(t_n) = 1$ for $n = 1, \dots, N-1$. Following the principle of dynamic programming, this is accomplished by choosing the withdrawal γ_n as

$$\gamma_n = \Gamma^V(t_n, \mathbf{Y}(t_n)) = \underset{\gamma \in \mathcal{A}}{\arg\max} \big\{ P_n(\gamma, \mathbf{Y}(t_n)) + V\left(t_n^+, \mathbf{X}(t_n^+ | \mathbf{X}(t_n), \gamma)\right) \big\} \tag{38}$$

in the admissible set $\mathcal{A} = \{\gamma : \gamma \geq 0, A(t_n^+ | \mathbf{X}(t_n), \gamma) \geq 0\}$. Here, we used $\mathbf{X}(t_n^+ | \mathbf{X}(t_n), \gamma)$ and $A(t_n^+ | \mathbf{X}(t_n), \gamma)$ to denote the state variables $\mathbf{X}$ and guarantee account balance A after withdrawal γ is made, given the value of the state variables $\mathbf{X}$ before the withdrawal. At any withdrawal time t_n the policyholder chooses the withdrawal $\gamma \in \mathcal{A}$ to maximize the sum of the payment $P_n(\gamma, \mathbf{Y}(t_n))$ received and the present value of the remaining term of the policy $V(t_n^+, \mathbf{X}(t_n^+))$. The strategy Γ^V given by (38) is called the *value maximization strategy*.

On the other hand, the optimization problem from the insurer's perspective considers the most unfavourable situation for the insurer. That is, by making suitable choices of γ_n's, the policyholder attempts to maximize the net initial liability function $L(0, \mathbf{X}(0))$. Even though a policyholder has little reason to pursue such a strategy, the fair fee rate under this strategy is guaranteed to cover the hedging cost of the GMWDB rider regardless of the withdrawal strategy of the policyholder (assuming the insurer can perfectly hedge the market risk). The withdrawal γ_n under $I(t_n) = 1$ for this strategy is given by

$$\gamma_n = \Gamma^L(t_n, \mathbf{Y}(t_n)) = \arg\max_{\gamma \in \mathcal{A}} \big\{ p_n(\gamma, \mathbf{Y}(t_n)) + L\left(t_n^+, \mathbf{X}(t_n^+ | \mathbf{X}(t_n), \gamma)\right) \big\}, \tag{39}$$

i.e., the sum of the payment made by the insurer and the net liability of the remaining term of the contract is maximized. The strategy Γ^L given by (39) is called the *liability maximization strategy*.

5.3. Implications

We now consider the two strategies in an idealized world where market risk can be completely hedged with neglegible market frictions as compared with the management fees. For simplicity, we avoid considering the mortality risk in this subsection by assuming zero mortality rates. As is mentioned in the previous subsection, assuming that the fair insurance fee follows from the liability maximization strategy, the insurer is guaranteed a nonnegative profit, regardless of the actual withdrawal strategies of the policyholder. If the policyholder behaves differently from this strategy, in particular, if he follows the value maximization strategy, the insurer generally makes a positive profit.

On the other hand, consider the situation where a middle agent purchases the GMWDB policy from the insuerer on behalf of the policyholder. The agent agrees to fulfill all the contract liabilities to the policyholder as implied by the policyholder's withdrawal behavior, but handles the withdrawals following his own optimal strategy. In particular, the agent follows the value maximization strategy when making withdrawals from the insurer, in the meantime hedge any market risk associated with the contract, e.g., through market index tracking ETFs at much lower costs than the typical management fees charged by a mutual fund manager. Then regardless of the policyholder's withdrawal behavior, the agent always makes a nonnegative profit. If the policyholder behaves differently from the value maximization strategy, he receives a value less than the maximal policy value received by the agent, who makes a positive profit out of the policyholder's sub-optimal behavior. Given that the agent carries out withdrawals that maximize the value rather than the cost to the insurer, the insurer in turn can afford to charge a less expensive fee than those implied by the liability maximization strategy, leading to more value for the policyholder. This seemingly win-win situation comes partly at the loss of the wealth manager, who now expect to receive less management fees. In this case the agent in effect reduces the market friction represented by the management fees by helping to improve the policyholder's value, and earns a profit.

6. Numerical Examples

To demonstrate the effect of management fees on the fair fees of GMWDB contracts under different withdrawal strategies, we carry out in this section several numerical experiments.

We investigate how the presence of management fees will lead to different fair fees for the two withdrawal strategies studied in previous sections under different market conditions and contract parameters.

6.1. Setup of the experiments

For illustration purposes, we assume a simple GMWDB contract as specified by (6), (9) as well as constant r, σ, α_{m} and α_{ins} so that the PDEs (20) and (29) hold, and set all mortality rates to zero for simplicity. We consider different contractual scenarios and calculate the fair fees implied by (21) under the withdrawal strategies given in Section 5.

It is assumed that the wealth and the guarantee accounts start at $W(0) = A(0) = 1$. The maturities of the contracts range from 5 to 20 years, with annual contractual withdrawals evenly distributed over the lifetime of the contracts. The first withdrawal occurs at the end of the first year and the last at the maturity. The management fee rate ranges from 0% up to 2%.

We consider several investment environments with the risk-free rate r at levels 1% and 5%, and the volatility of the index σ at 10% and 30%, to represent different market conditions such as low/high growth and low/high volatility scenarios. In addition, the penalty rate β may take values at 10% or 20%.

We compute the initial policy value $V(0, \mathbf{Y}(0))$ as well as the initial net liability $L(0, \mathbf{X}(0))$ at time 0 numerically by following Algorithms 1 and 2 simultaneously. The withdrawal strategies Γ^L and Γ^V are considered separately. The PDEs (20) and (29) are solved using Crank-Nicholson finite difference method (Crank and Nicolson [8], Hirsa [13]) with appropriate terminal and jump conditions for both functions under both strategies. This leads to the initial values and liabilities $V(0, \mathbf{X}(0); \Gamma^L)$, $L(0, \mathbf{X}(0); \Gamma^L)$, $V(0, \mathbf{X}(0); \Gamma^V)$ and $L(0, \mathbf{X}(0); \Gamma^V)$ under both strategies. Here, we made the dependence of these functions on the strategies explicit. The fair fee rates under both strategies were obtained by solving (21) using a standard root-finding numerical scheme.

6.2. Results and implications

The fair fees and corresponding total policy values are shown in Figures 1 and 2 for two market conditions: a low interest rate market with high volatility ($r = 1\%, \sigma = 30\%$) and a high interest rate market with low volatility ($r = 5\%, \sigma = 10\%$), respectively. Fair fee rates obtained for all market conditions and contract parameters and the corresponding policy values can be found in Tables 1 through 4.

We first observe from these numerical results that the fair fee rate implied by the liability maximization strategy is always higher, and the corresponding policyholder's total value always lower, than those implied by the value maximization strategy, unless management fees are absent, in which case these quantities are equal. These are to be expected from the definitions of the two strategies. We also note that under the market condition of low interest rate with high volatility, a much higher insurance fee rate is required than under the market condition of high interest rate with low volatility, for the obvious reason that under adverse market conditions, the guarantee is more valuable. Moreover, a higher penalty rate results in a lower insurance fee since a higher penalty rate discourages the policyholder from making more desirable withdrawals that exceed the contracted values.

Furthermore, the results show that under most market conditions or contract specifications, the fair insurance fee rate obtained is highly sensitive to the management fee rate regardless of the withdrawal strategies, as seen from Figures 1 and 2. In particular, the fair fee rate implied by the liability maximization strategy always increases with the management fee rate, since the management fees cause the wealth account to decrease, leading to higher liability for the insurer to fulfill. On the other hand, the fair fee rate implied by the value maximization strategy first increases then decreases with the management fee rate, since at high management fee rates, a rational policyholder tends to withdraw more and early to avoid the management fees, which in turn reduces the liability and generates more penalty incomes for the insurer.

A major insight from the numerical results is that with increasing management fees, the value maximization withdrawals of a rational policyholder deviates more from the liability maximization withdrawals assumed by the insurer. In particular, it is seen by examining Figures 1 and 2 that the fair fee rates implied by the two strategies differ more significantly under the following conditions:

- longer maturity T,
- lower penalty rate β,
- higher interest rate r, and
- higher management fee rate α_{m}.

Moreover, careful examination of results listed in Tables 1 and 2 reveals that the index volatility σ does not seem to contribute significantly to this discrepancy. These observations are intuitively reasonable: The contributors listed above all imply that the wealth manager's total value $M(0)$ will be higher. There are more incentives to withdraw early to achieve higher policy value in the form of reduced management fees. The corresponding differences between the policyholder's values follow similar patterns. Of particular interest is that in some cases, as shown in Figure 2, the fair fee rate implied by maximizing policyholder's value can become negative. This implies that the policyholder would want to withdraw more and early due to high management fees to such an extent, that the penalties incurred exceed the total value of the GMWDB rider. On the other hand, the fair fee rate implied by maximizing the liability is always positive.

7. Conclusions

Determining accurate hedging costs of VA guarantees is a significant issue for VA providers. While the effect of management fees on policyholder's withdrawal behaviors is typically ignored in the VA literature, it was demonstrated in this article that this effect on the pricing of GMWDB contract can be significant. As a form of market friction, management fees can affect policyholders' withdrawal behaviors, causing large deviations from the "optimal" (liability maximization) withdrawal behaviors often assumed in the literature.

Two different policyholder's withdrawal strategies were considered: liability maximization and value maximization when management fees are present. We demonstrated that these two withdrawal strategies imply different fair insurance fee rates, where maximizing policy value implies lower fair fees than those implied by maximizing liability, or equivalently, maximizing the "total value" of the contract, which represents the maximal hedging costs from the insurer's perspective.

We identify the difference between the initial investment plus the value of the guarantee and the total value of the policyholder as the wealth manager's total value, which causes the discrepancy between the two withdrawal strategies. We further identify a number of factors that contribute to this discrepancy through a series of illustrating numerical experiments. Our findings identify the management fees as a potential cause of discrepancy between the fair fee rates implied by the liability maximization strategy, often assumed from the insurer's perspective for VA pricing, and the prevailing market rates for VA contracts with GMWDB or similar riders.

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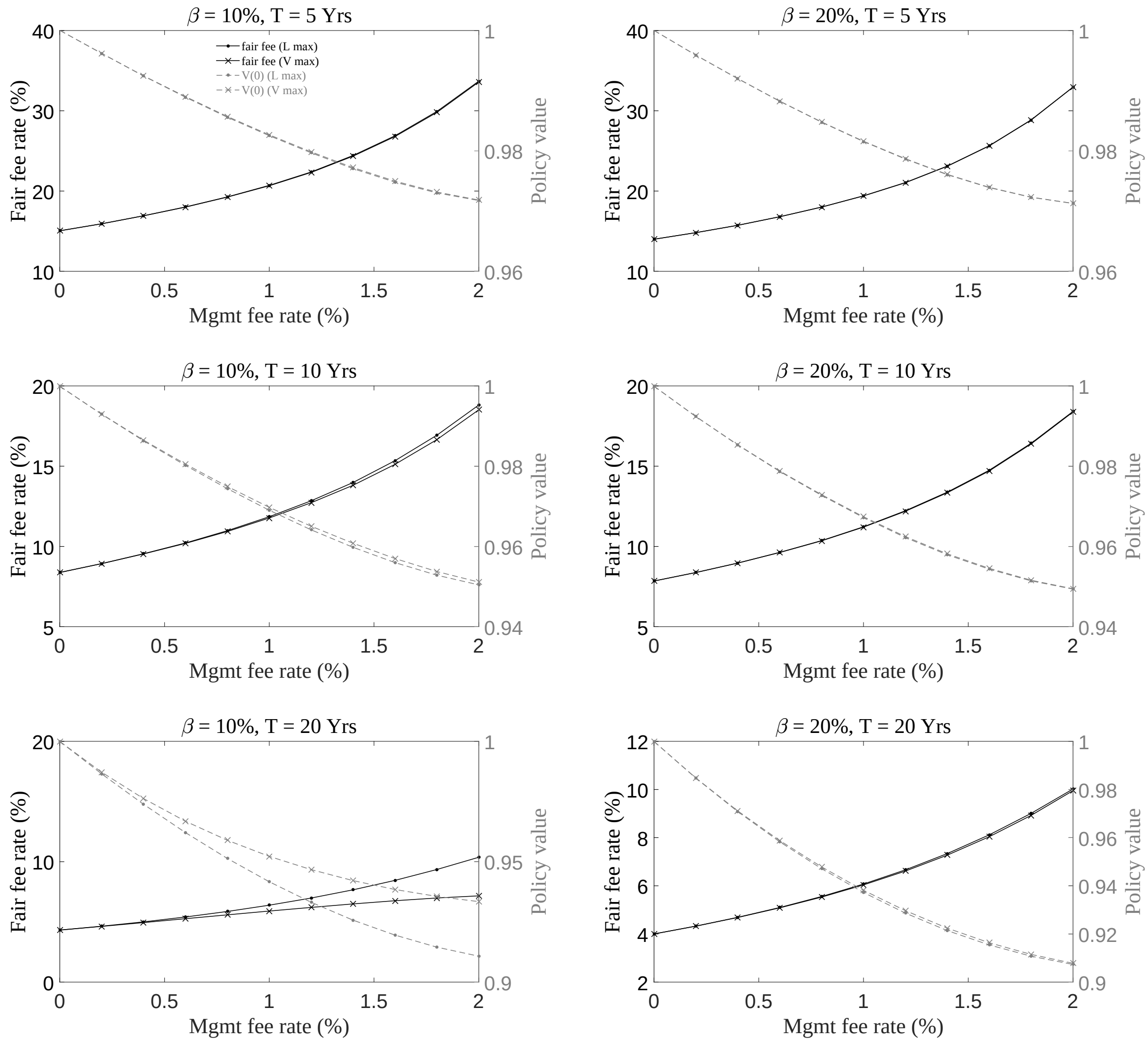

Figure 1: Fair insurance fee rates and policy values as a function of management fee rates α_{m} for risk-free rate $r = 1\%$ and volatility $\sigma = 30\%$, for penalty rates $\beta = 10\%\,,20\%$ and maturities $T = 5\,,10\,,20$ years. The left axis and dark plots refer to the fair fees in percentage; The right axis and gray plots refer to the policy values. Legends across all plots are shown in the upper left panel.

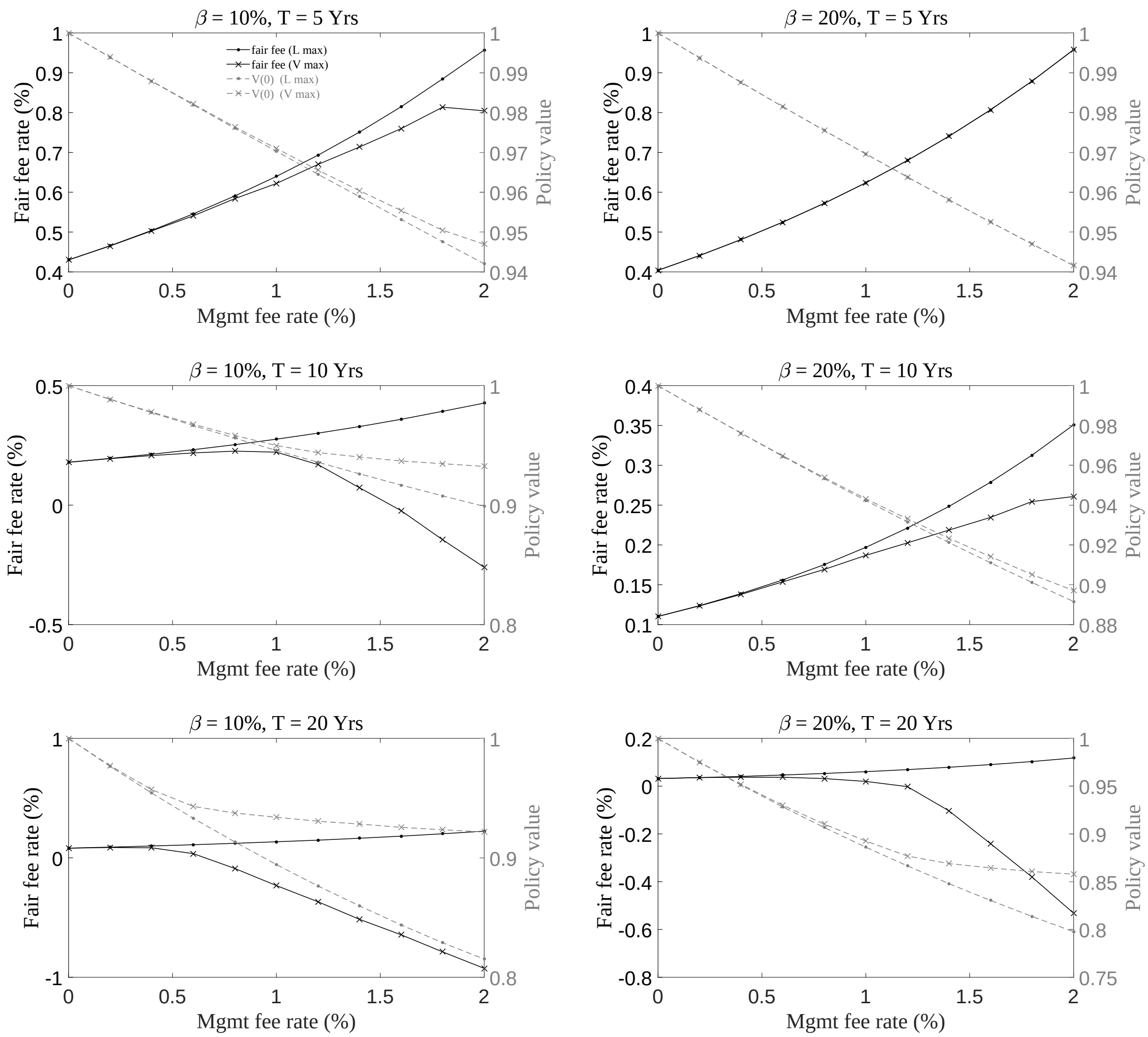

Figure 2: Fair insurance fee rates and policy values as a function of management fee rates α_{m} for risk-free rate $r = 5\%$ and volatility $\sigma = 10\%$, for penalty rates $\beta = 10\%\,,20\%$ and maturities $T = 5\,,10\,,20$ years. The left axis and dark plots refer to the fair fees in percentage; The right axis and gray plots refer to the policy values. Legends across all plots are shown in the upper left panel.

Table 1: Fair fee rate α_{ins} (%) based on the liability maximization strategy Γ^L.

Parameters				α_{m}										
r(%)	σ(%)	β(%)	T	0%	0.2%	0.4%	0.6%	0.8%	1%	1.2%	1.4%	1.6%	1.8%	2%
1	10	10	5	3.08	3.48	3.96	4.59	5.41	6.65	8.66	12.18	17.67	23.53	29.92
			10	1.66	1.92	2.25	2.66	3.21	3.97	5.08	6.77	9.26	12.17	15.31
			20	0.82	0.97	1.17	1.43	1.77	2.22	2.83	3.68	4.81	6.19	7.71
		20	5	3.08	3.47	3.95	4.55	5.34	6.48	8.39	12.01	17.66	23.55	29.92
			10	1.66	1.91	2.23	2.62	3.13	3.83	4.88	6.60	9.20	12.17	15.31
			20	0.81	0.96	1.16	1.40	1.71	2.13	2.72	3.57	4.74	6.17	7.71
	30	10	5	15.05	15.92	16.92	18.02	19.27	20.70	22.37	24.43	26.88	29.91	33.69
			10	8.38	8.93	9.55	10.22	10.98	11.85	12.84	13.98	15.34	16.93	18.81
			20	4.32	4.64	5.00	5.41	5.87	6.39	6.98	7.66	8.45	9.35	10.37
		20	5	13.99	14.82	15.73	16.80	18.01	19.40	21.06	23.11	25.64	28.86	32.95
			10	7.85	8.38	8.97	9.63	10.36	11.22	12.22	13.38	14.74	16.43	18.41
			20	4.00	4.33	4.69	5.10	5.55	6.07	6.66	7.34	8.11	9.01	10.02
5	10	10	5	0.43	0.47	0.50	0.55	0.59	0.64	0.69	0.75	0.81	0.88	0.96
			10	0.18	0.20	0.21	0.23	0.25	0.28	0.30	0.33	0.36	0.39	0.43
			20	0.08	0.09	0.10	0.11	0.12	0.13	0.15	0.16	0.18	0.20	0.22
		20	5	0.40	0.44	0.48	0.52	0.57	0.62	0.68	0.74	0.81	0.88	0.96
			10	0.11	0.12	0.14	0.16	0.18	0.20	0.22	0.25	0.28	0.31	0.35
			20	0.03	0.04	0.04	0.05	0.05	0.06	0.07	0.08	0.09	0.10	0.12
	30	10	5	5.33	5.48	5.65	5.81	5.99	6.17	6.35	6.55	6.75	6.96	7.17
			10	2.91	3.02	3.12	3.23	3.35	3.47	3.60	3.73	3.86	4.01	4.16
			20	1.58	1.65	1.74	1.82	1.91	2.00	2.10	2.21	2.32	2.43	2.55
		20	5	4.97	5.13	5.28	5.44	5.61	5.79	5.97	6.15	6.35	6.55	6.76
			10	2.27	2.35	2.43	2.52	2.61	2.71	2.81	2.91	3.02	3.13	3.25
			20	1.08	1.13	1.19	1.24	1.31	1.37	1.43	1.50	1.58	1.65	1.73

Table 2: Fair fee rate α_{ins} (%) based on the policy value maximization strategy Γ^V.

Parameters				α_{m}										
r(%)	σ(%)	β(%)	T	0%	0.2%	0.4%	0.6%	0.8%	1%	1.2%	1.4%	1.6%	1.8%	2%
1	10	10	5	3.08	3.47	3.96	4.57	5.36	6.57	8.55	12.12	17.66	23.55	29.93
			10	1.66	1.92	2.23	2.61	3.13	3.81	4.86	6.44	8.99	12.11	15.30
			20	0.82	0.96	1.09	1.21	1.33	1.44	1.50	1.59	1.65	1.70	1.81
		20	5	3.08	3.47	3.95	4.55	5.34	6.48	8.38	12.01	17.66	23.55	29.92
			10	1.66	1.91	2.23	2.62	3.12	3.80	4.84	6.56	9.19	12.17	15.31
			20	0.81	0.96	1.16	1.39	1.67	2.05	2.60	3.43	4.65	6.14	7.70
	30	10	5	15.05	15.92	16.91	18.00	19.25	20.66	22.32	24.34	26.79	29.80	33.58
			10	8.38	8.93	9.53	10.19	10.93	11.77	12.72	13.81	15.11	16.65	18.51
			20	4.32	4.63	4.94	5.27	5.59	5.90	6.20	6.49	6.76	6.99	7.16
		20	5	13.99	14.82	15.73	16.80	18.00	19.39	21.04	23.09	25.62	28.84	32.93
			10	7.85	8.38	8.96	9.62	10.35	11.20	12.19	13.34	14.70	16.38	18.37
			20	4.00	4.32	4.68	5.08	5.53	6.04	6.61	7.27	8.04	8.92	9.95
5	10	10	5	0.43	0.47	0.50	0.54	0.58	0.62	0.67	0.71	0.76	0.81	0.80
			10	0.18	0.19	0.21	0.22	0.23	0.22	0.17	0.07	-0.02	-0.14	-0.26
			20	0.08	0.09	0.08	0.04	-0.09	-0.23	-0.37	-0.51	-0.64	-0.79	-0.93
		20	5	0.40	0.44	0.48	0.52	0.57	0.62	0.68	0.74	0.81	0.88	0.96
			10	0.11	0.12	0.14	0.15	0.17	0.19	0.20	0.22	0.23	0.25	0.26
			20	0.03	0.04	0.04	0.04	0.03	0.02	-0.00	-0.10	-0.24	-0.38	-0.53
	30	10	5	5.33	5.48	5.64	5.79	5.95	6.11	6.27	6.43	6.60	6.75	6.94
			10	2.91	3.01	3.10	3.18	3.26	3.34	3.41	3.47	3.53	3.58	3.60
			20	1.58	1.64	1.68	1.72	1.74	1.75	1.75	1.74	1.74	1.73	1.71
		20	5	4.97	5.13	5.28	5.44	5.61	5.78	5.96	6.14	6.32	6.51	6.70
			10	2.27	2.35	2.43	2.51	2.59	2.67	2.74	2.82	2.88	2.94	3.00
			20	1.08	1.13	1.18	1.22	1.23	1.24	1.23	1.21	1.18	1.14	1.09

Table 3: Total policy value $V(0, \mathbf{X}(0); \Gamma^L)$ based on the liability maximization strategy Γ^L.

Parameters				α_m										
r(%)	σ(%)	β(%)	T	0%	0.2%	0.4%	0.6%	0.8%	1%	1.2%	1.4%	1.6%	1.8%	2%
1	10	10	5	1.00	0.99	0.99	0.98	0.98	0.98	0.97	0.97	0.97	0.97	0.97
			10	1.00	0.99	0.98	0.97	0.97	0.96	0.95	0.95	0.95	0.95	0.95
			20	1.00	0.98	0.96	0.95	0.94	0.92	0.92	0.91	0.90	0.90	0.90
		20	5	1.00	0.99	0.99	0.98	0.98	0.98	0.97	0.97	0.97	0.97	0.97
			10	1.00	0.99	0.98	0.97	0.96	0.96	0.95	0.95	0.95	0.95	0.95
			20	1.00	0.98	0.96	0.95	0.93	0.92	0.91	0.91	0.90	0.90	0.90
	30	10	5	1.00	1.00	0.99	0.99	0.99	0.98	0.98	0.98	0.97	0.97	0.97
			10	1.00	0.99	0.99	0.98	0.97	0.97	0.96	0.96	0.96	0.95	0.95
			20	1.00	0.99	0.97	0.96	0.95	0.94	0.93	0.93	0.92	0.91	0.91
		20	5	1.00	1.00	0.99	0.99	0.98	0.98	0.98	0.98	0.97	0.97	0.97
			10	1.00	0.99	0.99	0.98	0.97	0.97	0.96	0.96	0.95	0.95	0.95
			20	1.00	0.98	0.97	0.96	0.95	0.94	0.93	0.92	0.92	0.91	0.91
5	10	10	5	1.00	0.99	0.99	0.98	0.98	0.97	0.96	0.96	0.95	0.95	0.94
			10	1.00	0.99	0.98	0.97	0.96	0.95	0.94	0.93	0.92	0.91	0.90
			20	1.00	0.98	0.95	0.93	0.91	0.89	0.88	0.86	0.84	0.83	0.82
		20	5	1.00	0.99	0.99	0.98	0.98	0.97	0.96	0.96	0.95	0.95	0.94
			10	1.00	0.99	0.98	0.96	0.95	0.94	0.93	0.92	0.91	0.90	0.89
			20	1.00	0.97	0.95	0.93	0.91	0.89	0.87	0.85	0.83	0.81	0.80
	30	10	5	1.00	1.00	0.99	0.99	0.98	0.98	0.97	0.97	0.96	0.96	0.96
			10	1.00	0.99	0.98	0.98	0.97	0.96	0.95	0.95	0.94	0.93	0.93
			20	1.00	0.98	0.97	0.96	0.94	0.93	0.92	0.91	0.90	0.89	0.88
		20	5	1.00	0.99	0.99	0.98	0.98	0.97	0.97	0.96	0.96	0.95	0.95
			10	1.00	0.99	0.98	0.97	0.96	0.95	0.94	0.93	0.92	0.91	0.91
			20	1.00	0.98	0.96	0.94	0.92	0.90	0.89	0.87	0.86	0.84	0.83

Table 4: Total policy value $V(0, \mathbf{X}(0); \Gamma^V)$ based on the policy value maximization strategy Γ^V.

Parameters				α_{m}										
$r(\%)$	$\sigma(\%)$	$\beta(\%)$	T	0%	0.2%	0.4%	0.6%	0.8%	1%	1.2%	1.4%	1.6%	1.8%	2%
1	10	10	5	1.00	0.99	0.99	0.98	0.98	0.98	0.97	0.97	0.97	0.97	0.97
			10	1.00	0.99	0.98	0.97	0.97	0.96	0.95	0.95	0.95	0.95	0.95
			20	1.00	0.98	0.97	0.96	0.95	0.95	0.94	0.94	0.94	0.93	0.93
		20	5	1.00	0.99	0.99	0.98	0.98	0.98	0.97	0.97	0.97	0.97	0.97
			10	1.00	0.99	0.98	0.97	0.96	0.96	0.95	0.95	0.95	0.95	0.95
			20	1.00	0.98	0.96	0.95	0.93	0.92	0.91	0.91	0.90	0.90	0.90
	30	10	5	1.00	1.00	0.99	0.99	0.99	0.98	0.98	0.98	0.97	0.97	0.97
			10	1.00	0.99	0.99	0.98	0.97	0.97	0.97	0.96	0.96	0.95	0.95
			20	1.00	0.99	0.98	0.97	0.96	0.95	0.95	0.94	0.94	0.94	0.93
		20	5	1.00	1.00	0.99	0.99	0.98	0.98	0.98	0.98	0.97	0.97	0.97
			10	1.00	0.99	0.99	0.98	0.97	0.97	0.96	0.96	0.95	0.95	0.95
			20	1.00	0.98	0.97	0.96	0.95	0.94	0.93	0.92	0.92	0.91	0.91
5	10	10	5	1.00	0.99	0.99	0.98	0.98	0.97	0.97	0.96	0.96	0.95	0.95
			10	1.00	0.99	0.98	0.97	0.96	0.95	0.94	0.94	0.94	0.93	0.93
			20	1.00	0.98	0.96	0.94	0.94	0.93	0.93	0.93	0.93	0.92	0.92
		20	5	1.00	0.99	0.99	0.98	0.98	0.97	0.96	0.96	0.95	0.95	0.94
			10	1.00	0.99	0.98	0.96	0.95	0.94	0.93	0.92	0.91	0.91	0.90
			20	1.00	0.97	0.95	0.93	0.91	0.89	0.88	0.87	0.86	0.86	0.86
	30	10	5	1.00	1.00	0.99	0.99	0.98	0.98	0.97	0.97	0.97	0.96	0.96
			10	1.00	0.99	0.98	0.98	0.97	0.97	0.96	0.96	0.95	0.95	0.95
			20	1.00	0.99	0.97	0.97	0.96	0.95	0.94	0.94	0.94	0.93	0.93
		20	5	1.00	0.99	0.99	0.98	0.98	0.97	0.97	0.96	0.96	0.95	0.95
			10	1.00	0.99	0.98	0.97	0.96	0.95	0.94	0.94	0.93	0.92	0.92
			20	1.00	0.98	0.96	0.94	0.93	0.92	0.91	0.90	0.89	0.88	0.88

References

[1] Agapova, A., 2011. Conventional mutual index funds versus exchange-traded funds. Journal of Financial Markets 14 (2), 323 – 343.

[2] Bacinello, A. R., Ortu, F., 1996. Fixed income linked life insurance policies with minimum guarantees: Pricing models and numerical results. European Journal of Operational Research 91 (2), 235–249.

[3] Bauer, D., Kling, A., Russ, J., 2008. A universal pricing framework for guaranteed minimum benefits in variable annuities. ASTIN Bulletin 38 (2), 621–651.

[4] Bélanger, A. C., Forsyth, P. A., Labahn, G., 2009. Valuing guaranteed minimum death benefit clause with partial withdrawals. Applied Mathematical Finance 16 (6), 451–496.

[5] Chen, Z., Forsyth, P. A., 2008. A numerical scheme for the impulse control formulation for pricing variable annuities with a guaranteed minimum withdrawal benefit (gmwb). Numerische Mathematik 109 (4), 535–569.

[6] Chen, Z., Vetzal, K., Forsyth, P. A., 2008. The effect of modelling parameters on the value of GMWB guarantees. Insurance: Mathematics and Economics 43, 165–173.

[7] Cramer, E., Matson, P., Rubin, L., 2007. Common practices relating to fasb statement 133, accounting for derivative instruments and hedging activities as it relates to variable annuities with guaranteed benefits. In: Practice Note. American Academy of Actuaries.

[8] Crank, J., Nicolson, P., 1947. A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type. Mathematical Proceedings of the Cambridge Philosophical Society 43 (1), 50–67.

[9] Dai, M., Kuen Kwok, Y., Zong, J., 2008. Guaranteed minimum withdrawal benefit in variable annuities. Mathematical Finance 18 (4), 595–611.

[10] Delbaen, F., Schachermayer, W., 2006. The Mathematics of Arbitrage. Springer.

[11] Forsyth, P., Vetzal, K., 2014. An optimal stochastic control framework for determining the cost of hedging of variable annuities. Journal of Economic Dynamics and Control 44, 29–53.

[12] Fung, M. C., Ignatieva, K., Sherris, M., 2014. Systematic mortality risk: An analysis of guaranteed lifetime withdrawal benefits in variable annuities. Insurance: Mathematics and Economics 58 (1), 103–115.

[13] Hirsa, A., 2012. Computational Methods in Finance. Chapman and Hall/CRC Financial Mathematics Series.

[14] Ho, T. S. Y., Lee, S. B., Choi, Y. S., 2005. Practical considerations in managing variable annuities. doi:10.1.1.114.7023. Citeseer.

[15] Huang, Y. T., Kwok, Y. K., 2016. Regression-based Monte Carlo methods for stochastic control models: variable annuities with lifelong guarantees. Quantitative Finance 16 (6), 905–928.

[16] Hyndman, C. B., Wenger, M., 2014. Valuation perspectives and decompositions for variable annuities. Insurance: Mathematics and Economics 55, 283–290.

[17] Kalberer, T., Ravindran, K., 2009. Variable Annuities. Risk Books.

[18] Kling, A., F., R., Rub, J., 2011. The impact of stochastic volatility on pricing, hedging, and hedge efficiency of withdrawal benefit guarantees in variable annuities. ASTIN Bulletin 41 (2), 511–545.

[19] Kostovetsky, L., 2003. Index mutual funds and exchange-traded funds. The Journal of Portfolio Management 29 (4), 80–92.

[20] Ledlie, M. C., Corry, D. P., Finkelstein, G. S., Ritchie, A. J., Su, K., Wilson, D. C. E., 2008. Variable annuities. British Actuarial Journal 14 (2), 327–389.

[21] Luo, X., Shevchenko, P. V., 2015. Fast numerical method for pricing of variable annuities with guaranteed minimum withdrawal benefit under optimal withdrawal strategy. International Journal of Financial Engineering 02 (03), 1550024:1–1550024:26.

[22] Luo, X., Shevchenko, P. V., 2015. Valuation of variable annuities with guaranteed minimum withdrawal and death benefits via stochastic control optimization. Insurance: Mathematics and Economics 62, 5–15.

[23] Milevsky, M. A., Salisbury, T. S., 2006. Financial valuation of guaranteed minimum withdrawal benefits. Insurance: Mathematics and Economics 38 (1), 21–38.

[24] Moenig, T., Bauer, D., 2015. Revisiting the risk-neutral approach to optimal policyholder behavior: A study of withdrawal guarantees in variable annuities. Review of Finance 20 (2), 759–794.

[25] Platen, E., Heath, D., 2006. A Benchmark Approach to Quantitative Finance. Springer.

[26] Poterba, J. M., Shoven, J. B., 2002. Exchange traded funds: A new investment option for taxable investors. Tech. rep., National bureau of economic research.

[27] Shevchenko, P. V., Luo, X., 2016. A unified pricing of variable annuity guarantees under the optimal stochastic control framework. Risks 4 (3), 22:1–22:31.

[28] Shevchenko, P. V., Luo, X., 2017. Valuation of variable annuities with guaranteed minimum withdrawal benefit under stochastic interest rate. Insurance: Mathematics and Economics 76 (Supplement C), 104 – 117.

[29] Sun, P., Kelkar, R., Dai, J., Huang, V., 2016. How effective is variable annuity guarantee hedging? Milliman Research Report: `http://au.milliman.com/insight/2016/How-effective-is-variable-annuity-guarantee-hedging`.